A class of (ℓ-dependent) potentials with the same number of (ℓ-wave) bound states

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Abstract

We introduce and investigate the class of central potentials

$$V_{\text{CIC}}(g^2, \mu^2, \ell, R; r) = -\frac{g^2}{R^2} \left(\frac{r}{R}\right)^{4\ell} \left\{ \left[1 + \left(\frac{1}{2\ell+1}\right) \left(\frac{r}{R}\right)^{2\ell+1}\right]^2 - 1 + \mu^2 \right\}^{-2},$$

which possess, in the context of nonrelativistic quantum mechanics, a number of ℓ-wave bound states given by the (ℓ-independent !) formula

$$N_\ell^{(\text{CIC})}\left(g^2, \mu^2\right) = \left\{\left\{\frac{1}{\pi}\sqrt{g^2 + \mu^2 - 1}\left(\sqrt{\mu^2 - 1}\right)^{-1} \arctan\left(\sqrt{\mu^2 - 1}\right)\right\}\right\}.$$

Here g and μ are two arbitrary real parameters, ℓ is the angular momentum quantum number, and the double braces denote of course the integer part. An extension of this class features potentials that possess the same number of ℓ-wave bound states and behave as $(a/r)^2$ both at the origin ($r \to 0^+$) and at infinity ($r \to \infty$), where a is an additional free parameter.

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Recently we revisited [1,2] the classical problem of establishing upper and lower limits for the number N_ℓ of ℓ-wave bound states possessed by a central

potential $V(r)$ (vanishing as $r \to \infty$) in the context of nonrelativistic quantum mechanics. As it is well known (see, for instance, [3]), this number N_ℓ coincides with the number of zeros, in the interval $0 < r < \infty$, of the zero-energy radial wave function $u_\ell(r)$, namely of the solution $u_\ell(r)$, characterized by the boundary condition $u_\ell(0) = 0$, of the zero-energy (stationary, radial) Schrödinger equation

$$-u_\ell''(r) + \left[V(r) + \frac{\ell(\ell+1)}{r^2} \right] u_\ell(r) = 0. \tag{1}$$

[Here and throughout we use of course the standard quantum-mechanical units such that $\hbar = 2m = 1$, where m is the mass of the particle under consideration, and ℓ (a nonnegative integer, $\ell = 0, 1, 2, \ldots$) is the angular momentum quantum number]. In this context we discovered a class of (ℓ-dependent) potentials, which (depend on two dimensionless parameters and) have the remarkable property to possess a number N_ℓ of ℓ-wave bound states that (is given by a neat formula and) does not depend on the angular momentum quantum number ℓ. These potentials, to which we assigned the name $V_{\mathrm{CIC}}(r)$ because we found them while visiting the Centro Internacional de Ciencias (CIC) in Cuernavaca (see Acknowledgements), read as follows:

$$V_{\mathrm{CIC}}(g^2, \mu^2, \ell, R; r) \equiv \frac{g^2}{R^2} f_{\mathrm{CIC}}\left(\mu^2, \ell; \frac{r}{R}\right), \tag{2a}$$

$$f_{\mathrm{CIC}}\left(\mu^2, \ell; x\right) = -x^{4\ell} \left[\left(1 + \frac{x^{2\ell+1}}{2\ell+1} \right)^2 - 1 + \mu^2 \right]^{-2}. \tag{2b}$$

Here and throughout R, g^2, μ^2 are three arbitrary positive parameters, the first of which has the dimensions of a length, and the other two are dimensionless and must satisfy the inequality $g^2 + \mu^2 > 1$ (which is necessary for the existence of bound states). The corresponding number $N_\ell^{(\mathrm{CIC})}$ of ℓ-wave bound states is indeed given by the following (ℓ-independent !) neat formula (proven below):

$$N_\ell^{(\mathrm{CIC})}\left(g^2, \mu^2\right) = \left\{\left\{\tilde{N}_\ell^{(\mathrm{CIC})}\left(g^2, \mu^2\right)\right\}\right\}, \tag{3}$$

where the double braces denote of course the integer part and

$$\tilde{N}_\ell^{(\mathrm{CIC})}\left(g^2, \mu^2\right) = \frac{1}{2\pi}\sqrt{g^2 + \mu^2 - 1}\left(\sqrt{1-\mu^2}\right)^{-1}\log\left[\frac{1+\sqrt{1-\mu^2}}{1-\sqrt{1-\mu^2}}\right] \tag{4a}$$
$$\text{if} \quad \mu^2 \leq 1,$$

$$\tilde{N}_\ell^{(\mathrm{CIC})}\left(g^2, \mu^2\right) = \frac{g}{\pi} \quad \text{if} \quad \mu^2 = 1, \tag{4b}$$

2

$$\tilde{N}_\ell^{(\mathrm{CIC})}\left(g^2,\mu^2\right) = \frac{1}{\pi}\sqrt{g^2+\mu^2-1}\left(\sqrt{\mu^2-1}\right)^{-1}\arctan\left(\sqrt{\mu^2-1}\right) \quad (4c)$$
$$\text{if} \quad \mu^2 \geq 1.$$

This remarkable property evokes a mathematical and pedagogical interest, and it is moreover of some applicative relevance because these potentials can be used as comparison potentials: clearly any potential $V(r)$ that is more "attractive" than an ℓ-wave CIC potential, see (2), $V(r) \leq V_{\mathrm{CIC}}(g^2,\mu^2,\ell,R;r)$ for $0 \leq r < \infty$, shall possess a number N_ℓ of ℓ -wave bound states at least as large as $N_\ell^{(\mathrm{CIC})}\left(g^2,\mu^2\right)$, $N_\ell \geq N_\ell^{(\mathrm{CIC})}\left(g^2,\mu^2\right)$, and conversely any potential $V(r)$ that is less "attractive" than an ℓ-wave CIC potential, see (2), $V(r) \geq V_{\mathrm{CIC}}(g^2,\mu^2,\ell,R;r)$ for $0 \leq r < \infty$, shall possess a number N_ℓ of ℓ-wave bound states not larger than $N_\ell^{(\mathrm{CIC})}\left(g^2,\mu^2\right)$, $N_\ell \leq N_\ell^{(\mathrm{CIC})}\left(g^2,\mu^2\right)$ [2]. This motivated us to write the present paper, in order to advertise this finding and to elaborate on it by providing some information on the shape $f_{\mathrm{CIC}}\left(\mu^2,\ell;x\right)$ of these potentials, see (2b).

First of all let us note the following qualitative features. Clearly this function is negative, $f_{\mathrm{CIC}}\left(\mu^2,\ell;x\right) < 0$, for all positive values of x, $0 < x < \infty$, and it vanishes proportionally to the inverse $[4\left(\ell+1\right)]$-power of x at large x,

$$\lim_{x\to\infty}\left[x^{4(\ell+1)}f_{\mathrm{CIC}}\left(\mu^2,\ell;x\right)\right] = -\left(2\ell+1\right)^4. \quad (5)$$

For $\ell = 0$, this function has its minimum at $x = 0$,

$$\min_{0\leq x<\infty}\left[f_{\mathrm{CIC}}\left(\mu^2,0;x\right)\right] = f_{\mathrm{CIC}}\left(\mu^2,0;0\right) = -\mu^{-4}, \quad (6)$$

and it increases monotonically from this minimum value to its vanishing value at $x = \infty$ (namely, $f_{\mathrm{CIC}}'\left(\mu^2,0;x\right) > 0$ for $0 < x < \infty$; more specifically, if $0 < \mu^2 < 6$ the second derivative of $f_{\mathrm{CIC}}\left(\mu^2,0;x\right)$ is everywhere negative, $f_{\mathrm{CIC}}''\left(\mu^2,0;x\right) < 0$ for $0 < x < \infty$, while if $\mu^2 > 6$, the second derivative changes sign, namely $f_{\mathrm{CIC}}''\left(\mu^2,0;x\right) > 0$ for $0 < x < \tilde{x}(\mu^2)$, $f_{\mathrm{CIC}}''\left(\mu^2,0;x\right) < 0$ for $\tilde{x}(\mu^2) < x < \infty$, with $\tilde{x}(\mu^2) = \sqrt{\frac{\mu^2-1}{5}} - 1$).

For positive ℓ, $\ell = 1, 2, \ldots$, this function vanishes at the origin proportionally to $x^{4\ell}$ and it has a single minimum at $x = x_{\min}\left(\mu^2,\ell\right)$,

$$x_{\min}\left(\mu^2,\ell\right) = \left[\frac{(2\ell+1)}{2\left(\ell+1\right)}\xi(\mu^2)\right]^{\frac{1}{2\ell+1}}, \quad (7)$$

where it attains the value

3

$$\min_{0 \leq x < \infty} \left[f_{\mathrm{CIC}} \left(\mu^2, \ell; x \right) \right] = f_{\mathrm{CIC}} \left[\mu^2, \ell; x_{\min} \left(\mu^2, \ell \right) \right]$$

$$= - \left(\ell + 1 \right)^2 \left[\frac{2(\ell+1)}{2\ell+1} \right]^{\frac{2}{2\ell+1}} \frac{[\xi(\mu^2)]^{\frac{4\ell}{2\ell+1}}}{[\xi(\mu^2) + 2\mu^2(\ell+1)]^2} \tag{8}$$

Here we use the short-hand notation $\xi \left(\mu^2 \right) = \sqrt{1 + 4\ell \left(\ell + 1 \right) \mu^2} - 1$. Note that $x_{\min} \left(\mu^2, \infty \right) = 1$, and $x_{\min} \left(\mu^2, \ell \right)$ grows as $\left[\frac{\ell (2\ell+1)^2 \mu^2}{(\ell+1)} \right]^{\frac{1}{2(2\ell+1)}}$ at large μ; while the minimum of $f_{\mathrm{CIC}} \left(\mu^2, \ell; x \right)$, see (8), decreases as $-\ell^2 \left(1 + \mu \right)^{-2}$ at large ℓ, and increases as $-L(\ell) \mu^{\frac{-4(\ell+1)}{2\ell+1}}$ at large μ with $L(\ell) = \left(\ell \right)^{\frac{2\ell}{2\ell+1}} \left(\ell + 1 \right)^{\frac{2(\ell+1)}{2\ell+1}} \left(2\ell + 1 \right)^{\frac{-2}{2\ell+1}}$.

The formulas (7) and (8), as well of course as the expression of the number of bound states, see (4b), simplify considerably for $\mu^2 = 1$:

$$x_{\min} \left(1, \ell \right) = \left[\frac{\ell \left(2\ell + 1 \right)}{\left(\ell + 1 \right)} \right]^{\frac{1}{2\ell+1}}, \tag{9}$$

$$\min_{0 \leq x < \infty} \left[f_{\mathrm{CIC}} \left(1, \ell; x \right) \right] = f_{\mathrm{CIC}} \left[1, \ell; x_{\min} \left(1, \ell \right) \right] = -\ell^{\frac{4\ell}{2\ell+1}} \left(\frac{\ell + 1}{2\ell + 1} \right)^{\frac{4(\ell+1)}{2\ell+1}}. \tag{10}$$

Another remarkable property of the function $f_{\mathrm{CIC}} \left(\mu^2, \ell; x \right)$ is displayed by the formula (that can be easily verified by direct integration)

$$\int_0^\infty dx \, \sqrt{- f_{\mathrm{CIC}} \left(\mu^2, \ell; x \right)} = \frac{\pi \, g}{\sqrt{g^2 + \mu^2 - 1}} \, \tilde{N}_\ell^{(\mathrm{CIC})} \left(g^2, \mu^2 \right), \tag{11}$$

see (4).

Graphs of the function $f_{\mathrm{CIC}} \left(\mu^2, \ell; x \right)$ (appropriately renormalized) are presented in Figs. 1 and 2 for $\ell = 0$ and $\ell = 2$ and for various values of μ.

Let us now prove the claim made above, namely the validity of (3) with (4). This is done (albeit without providing here any explanation of how this result was discovered [2]) by relating as follows the solution $u_\ell(r)$ of the zero-energy radial Schrödinger equation (1) (or rather its logarithmic derivative) to a "phase function" $\eta_\ell(r)$ that provides a convenient tool to count the zeros of $u_\ell(r)$:

$$\sqrt{-V(r)} \cot \left[\eta_\ell(r) \right] - \frac{V'(r)}{4 \, V(r)} = \frac{u_\ell'(r)}{u_\ell(r)}. \tag{12}$$

It is indeed plain that the boundary condition $u_\ell(0) = 0$ entails $\eta_\ell(0) = 0$, while the definition $\eta_\ell(\infty) = \tilde{N}_\ell \pi$ entails that the number N_ℓ of zeros of $u_\ell(r)$ in $0 < r < \infty$ is given by the integer part of $\tilde{N}_\ell$. It is moreover easy to verify
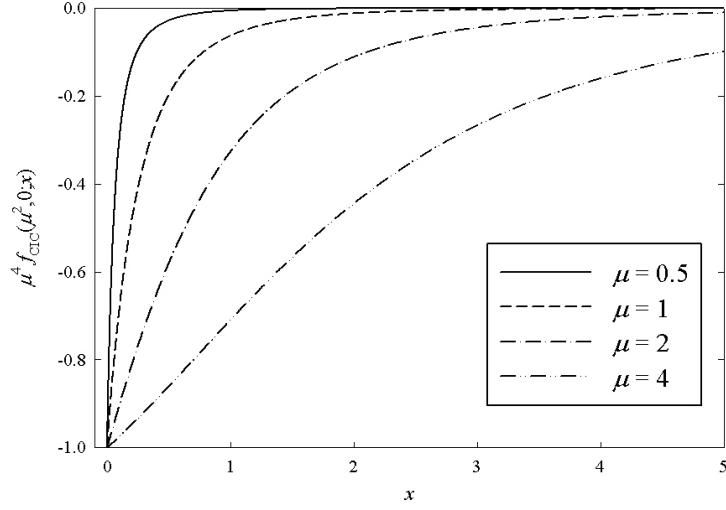
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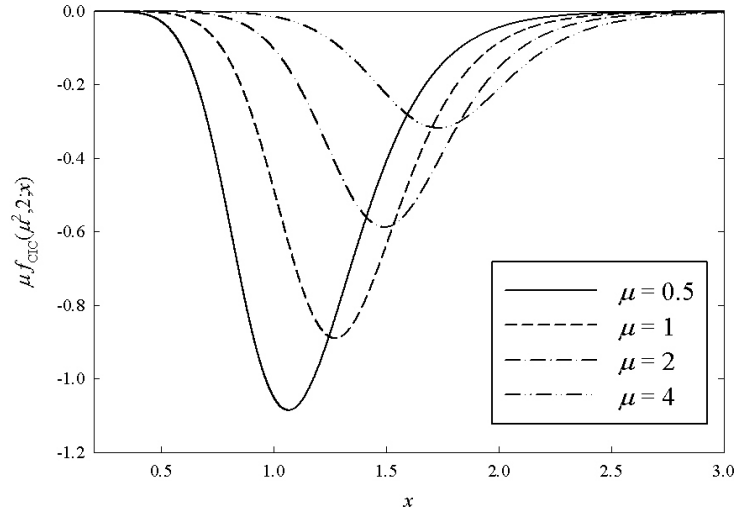Fig. 1. Graphs of the function $\mu^4 f_{\mathrm{CIC}}\left(\mu^2, 0; x\right)$, see (2b), for four values of μ.

Fig. 2. Graphs of the function $\mu\, f_{\mathrm{CIC}}\left(\mu^2, 2; x\right)$, see (2b), for four values of μ.

that the zero-energy radial Schrödinger equation (1) yields for $\eta_\ell(r)$ the simple first-order ODE

$$\eta_\ell'(r) \left\{1 + \frac{\mu^2 - 1}{g^2} \left[\sin\, \eta_\ell(r)\right]^2\right\}^{-1} = \sqrt{-V(r)}, \tag{13}$$

provided the potential $V(r)$ satisfies the nonlinear ODE

$$-\frac{\ell\,(\ell+1)}{r^2} + \frac{5}{16}\left(\frac{V'(r)}{V(r)}\right)^2 - \frac{V''(r)}{4V(r)} + \frac{\mu^2 - 1}{g^2}\, V(r) = 0. \tag{14}$$

5

And it is as well easy to verify that the potential $V_{\mathrm{CIC}}(g^2, \mu^2, \ell, R; r)$, see (2), does indeed satisfy this ODE, (14).

It is then plain, by taking advantage of the boundary values of $\eta_\ell(r)$ at $r = 0$ and at $r = \infty$ (see above) and of the relation (11) with (2), that the integration from $r = 0$ to $r = \infty$ of the ODE (13) yields the expression (4) for $\tilde{N}_\ell^{(\mathrm{CIC})}$. Q.E.D.

Finally let us point out a consequence of the obvious identity of the zero-energy Schrödinger equation which obtains from (1) by replacing in it $V(r)$ with $W(r)$, with the zero-energy Schrödinger equation which obtains from (1) by replacing in it ℓ with λ, provided

$$W(r) = \left(\frac{a}{r}\right)^2 + V(r), \tag{15}$$

$$\lambda \equiv \lambda\left(a^2, \ell\right) = -\frac{1}{2} + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + a^2}. \tag{16}$$

This of course entails that the number N_ℓ of ℓ-wave bound states possessed by the potential

$$W_{\mathrm{CIC}}(g^2, \mu^2, a^2, \ell, R; r) \equiv \frac{1}{R^2} F_{\mathrm{CIC}}\left(g^2, \mu^2, a^2, \ell; \frac{r}{R}\right) \tag{17a}$$

$$F_{\mathrm{CIC}}\left(g^2, \mu^2, a^2, \ell; x\right) = \left(\frac{a}{x}\right)^2 + g^2 f_{\mathrm{CIC}}(\mu^2, \lambda; x), \tag{17b}$$

where a is an arbitrary real constant and $f_{\mathrm{CIC}}(\mu^2, \lambda; x)$ is of course defined by (2b) with (16), is still given by (3) with (4) (this formula provides now the number of bound-states with angular momentum quantum number λ rather than ℓ; but this number does not depend on this quantum number, see (4); which of course also entails that λ is not required here to be an integer). This remark entails that the potential $W_{\mathrm{CIC}}(g^2, \mu^2, a^2, \ell, R; r)$, which depends now on the *three* dimensionless (arbitrary, positive) constants g^2, μ^2, a^2, provides an additional, more flexible tool to assess, by comparison techniques, the number N_ℓ of ℓ-wave bound states possessed by a given central potential $V(r)$. Graphs of the function $F_{\mathrm{CIC}}\left(g^2, \mu^2, a^2, \ell; x\right)$ are presented in Figs. 3, 4 and 5 for $g = 10$, $\mu = 1$, $\ell = 0$, $\ell = 2$ and $\ell = 10$ and for various values of a. In Fig. 6, graphs of the function $F_{\mathrm{CIC}}\left(g^2, \mu^2, a^2, \ell; x\right)$ are presented for $a = 1$, $\ell = 5$ and for various values of g and μ such as the number of ℓ-wave bound states is always equal to three, see (4).

Let us end this paper by emphasizing that, because of the availability of two, or even three, free parameters (for every given value of the angular momentum quantum number ℓ), the family of CIC potentials considered herein is quite flexible (as also shown by the figures); these makes these potentials –
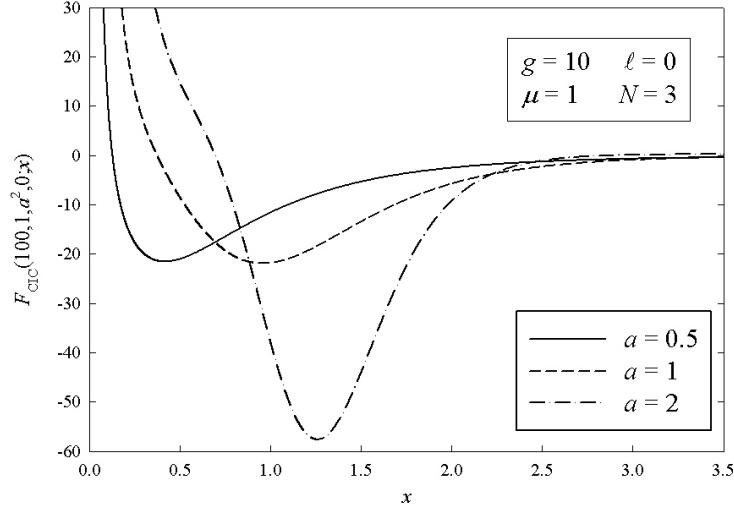
Fig. 3. Graphs of the function $F_{\text{CIC}}\left(100, 1, a^2, 0; x\right)$, see (17b), for three values of a. All these potentials, see (17a), possess three S-wave bound states, see (4b).
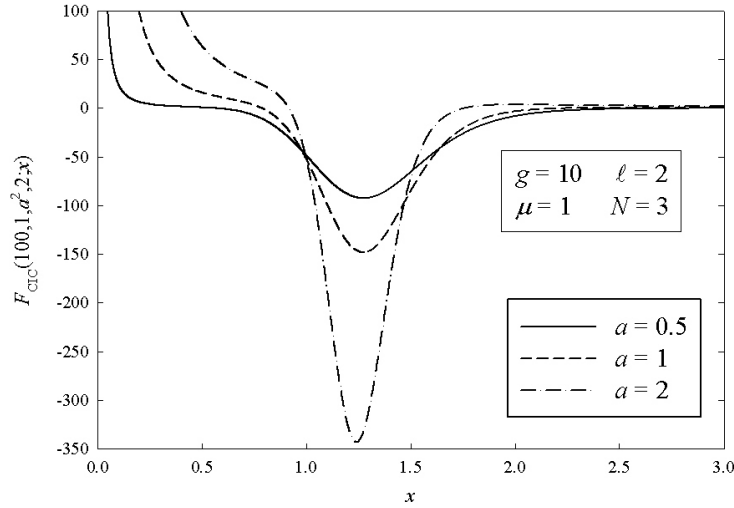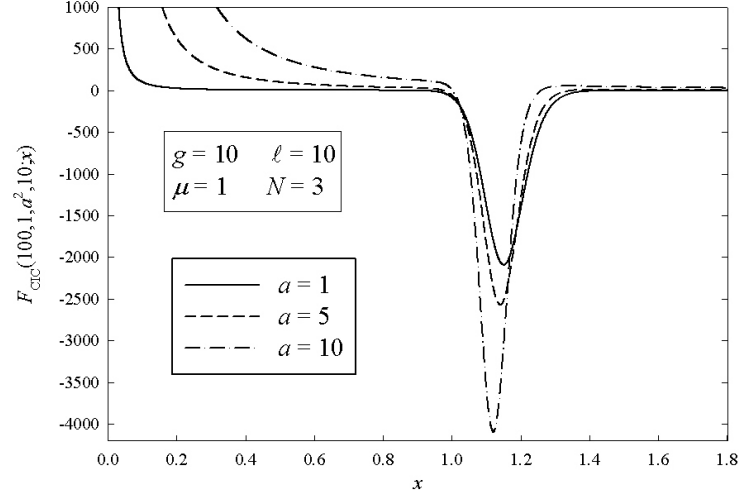

Fig. 4. Graphs of the function $F_{\text{CIC}}\left(100, 1, a^2, 2; x\right)$, see (17b), for three values of a. All these potentials, see (17a), possess three D-wave bound states, see (4b).

which have a known number of bound states – quite convenient to be used as comparison potentials, whenever one wishes to assess quickly and easily the number of (ℓ-wave) bound states associated with a given potential.

Acknowledgments

Fig. 5. Graphs of the function $F_{\text{CIC}}\left(100, 1, a^2, 10; x\right)$, see (17b), for three values of a. All these potentials, see (17a), possess three ℓ-wave bound states with $\ell = 10$, see (4b).
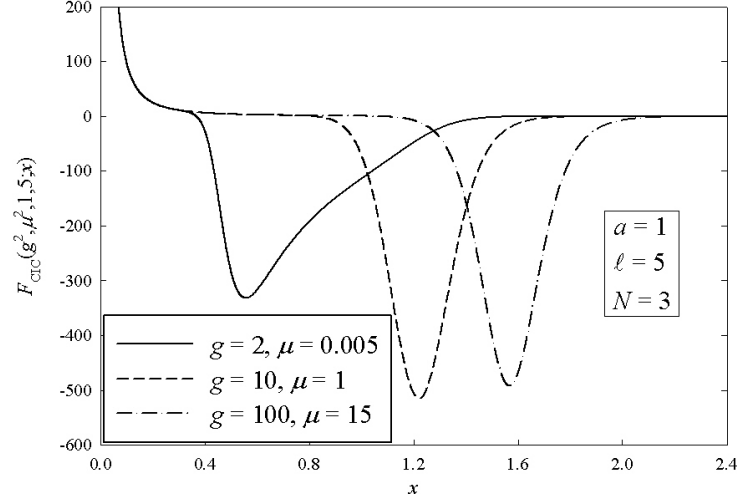

Fig. 6. Graphs of the function $F_{\text{CIC}}\left(g^2, \mu^2, 1, 5; x\right)$, see (17b), for three values of g and μ possessing three ℓ-wave bound states with $\ell = 5$, see (4).

8

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